Characterising the set of (untruncated) signature

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The starting point...

Hambly, B. and Lyons, T., 2010. Uniqueness for the signature of a path of bounded variation and the reduced path group. Annals of Mathematics, pp.109-167.

Paths with concatenation

Definition (Paths)

The set of paths:

$$P = \left\{ x : [0,1] \to \mathbb{R}^d : x \text{ continuous}, x(0) = 0 \right\}.$$

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Given $x, y \in P$, define

$$(xy)(t) = \begin{cases} x(2t), & t \in \left[0, \frac{1}{2}\right]; \\ x(1) + y(2t-1), & t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

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Goal:

Turn $(P, \ )$ into a group.

Definition of a group

Definition

Recall $(G,*)$ is a group if

1. (Closure)
$$x*y\in G \qquad \forall x,y\in G;$$
2. (Associativity)
$$(x*y)*z=x*(y*z), \qquad \forall x,y,z\in G.$$
3. (Identity)
$$\exists e\in G \qquad x*e=e*x=x \quad \forall x\in G.$$
4. **(Inverse)**
$$\forall x\in G \qquad \exists y\in G \qquad x*y=e$$

No inverse

Fact

Path almost never have inverse wrt concatenation.
Identity element:

$$e(t) = 0 \; \forall t \in [0,1].$$

Candidate for inverse:

$$\overleftarrow{x}(t) = x(1-t) - x(1).$$

Then

$$(x\overleftarrow{x})(t) = \begin{cases} x(2t), & t \in \left[0, \frac{1}{2}\right]; \\ x(2-2t), & t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

Unless $x = e$,

$$x\overleftarrow{x} \neq e.$$

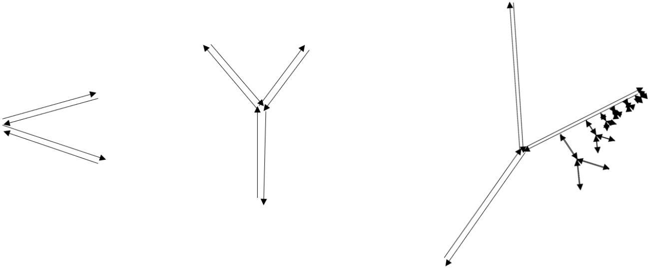
No inverse

Need an equivalence relation $\sim$ on P such that:

1

$$x\overleftarrow{x} \sim \overleftarrow{x}x \sim e$$

All of the followings are $\sim e$:



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Need an equivalence relation $\sim$ on P such that:

1. $$x\overleftarrow{x} \sim \overleftarrow{x}x \sim e$$
2. $$x_1 \sim x_2, y_1 \sim y_2 \implies x_1 y_1 \sim x_2 y_2.$$

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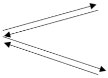
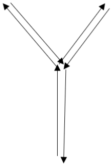
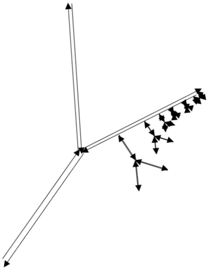






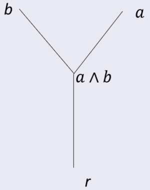
Tree-like path

Definition ($\mathbb{R}$-tree, Favre-Jonsson04)

If τ is a partially ordered set satisfying:
1.τ has global min. (call r) and pairwise min.
2. For any $t \in \tau$, the set below is totally ordered

$$\{s \in \tau : s \preceq t\};$$

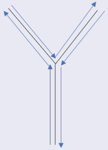
3. $\exists L : \tau \to \mathbb{R}_{\geq 0}$ map intervals in τ bijectively to intervals in $\mathbb{R}$.
If $d(s,t) = L(t) + L(s) - 2L(s \wedge t)$,
then (τ, d) is a $\mathbb{R}$-tree.



Tree-like path

Definition (Loop in a tree)

Let τ be a $\mathbb{R}$-tree. A loop in τ is a continuous function $\phi : [0, 1] \to \tau$ such that $\phi(0) = \phi(1)$.



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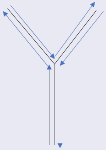

Definition (Tree-like Hambly-Lyons10)

A path $x : [0,1] \to V$ is tree-like if there exists:

1. a loop ϕ in a $\mathbb{R}$-tree τ;
2. a continuous function $\psi : \tau \to V$

$$x = \psi \circ \phi.$$

Tree-like equivalence

Definition (Tree-like relation, Hambly-Lyons10)

We define a relation $\sim$ on paths by $x_1 \sim x_2$ if

$$x_1 \overleftarrow{x_2} \qquad \text{is a tree-like path.}$$

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Theorem (Hambly-Lyons10)

*The relation $\sim$ is an **equivalence** relation on the set of bounded variation paths.*
The set

$$RP = \left\{ [x]_\sim : x : [0,1] \to \mathbb{R}^d \ \textit{continuous, } BV, x(0) = 0 \right\}$$

is called the (BV) Reduced path group.

Proof of Tree-like equivalence 1

Definition (Signature)

The signature of $x : [0,1] \to \mathbb{R}^d$ is defined by

$$S(x)_{0,1} = 1 + \int_0^1 \mathrm{d}x_{t_1} + \ldots + \int_{0<t_1<\ldots<t_n<1} \mathrm{d}x_{t_1} \otimes \ldots \otimes \mathrm{d}x_{t_n} + \ldots .$$

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If x has bounded variation (BV),

$$S(x)_{0,1} = 1 \iff x \text{ is tree-like.}$$

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Corollary

$x_1 \sim x_2 \iff S(x_1)_{0,1} = S(x_2)_{0,1}$ *for BV paths.*

Reduced paths

Definition (Reduced paths)

A path $x : [0,1] \to \mathbb{R}^d$ is **reduced** if

$$\text{length}(x) = \inf \left\{ \text{length}(\alpha) : S(\alpha)_{0,1} = S(x)_{0,1} \right\}.$$

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Corollary (Hambly-Lyons10)

If x_1, x_2 are reduced paths with bounded variation,

$$S(x_1)_{0,1} = S(x_2)_{0,1} \iff x_1 = x_2 \textit{ up to translation, reparam.}$$

Proof of uniqueness

Definition

Let $Sh(n,k)$ be the set of bijections $\sigma : \{1,\ldots,n+k\} \to \{1,\ldots,n+k\}$ such that

$$\sigma(1) < \sigma(2) < \ldots < \sigma(n)$$
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Lemma ("Shuffle product formula")

If $x=\left(x^{1},\ldots,x^{d}\right)$,

$$\int_{0<t_1<\ldots<t_n<t}\mathrm{d}x_{t_1}^{i_1}\ldots\mathrm{d}x_{t_n}^{i_n}\cdot\int_{0<t_1<\ldots<t_k<t}\mathrm{d}x_{t_1}^{i_{n+1}}\ldots\mathrm{d}x_{t_k}^{i_{n+k}}$$
$$=\sum_{\sigma\in Sh(n,k)}\int_{0<t_1<\ldots<t_{n+k}<t}\mathrm{d}x_{t_1}^{i_{\sigma^{-1}(1)}}\ldots\mathrm{d}x_{t_{n+k}}^{i_{\sigma^{-1}(n+k)}}.$$

Linear functionals on signatures

Examples

Closure under products

$$\left(\int_{0<t_1<t_2<t} dx^1_{t_1} dx^2_{t_2}\right)\left(\int_{0<t_1<t} dx^1_{t_1}\right)$$
$$= 2\int_{0<t_1<t_2<t_3<t} dx^1_{t_1} dx^1_{t_2} dx^2_{t_3} + \int_{0<t_1<t_2<t_3<t} dx^1_{t_1} dx^2_{t_2} dx^1_{t_3}.$$

Closure under integration

$$\int_0^t \left(\int_{0<t_1<t_2<u} dx^1_{t_1} dx^2_{t_2} + \int_{0<t_1<u} dx^1_{t_1}\right) dx^2_u$$
$$= \int_{0<t_1<t_2<t_3<u} dx^1_{t_1} dx^2_{t_2} dx^2_{t_3} + \int_{0<t_1<t_2<u} dx^1_{t_1} dx^2_{t_2}.$$

Proof

Lemma (B. Geng Lyons Yang 16)

If $t \to S(x)_{0,t}$ and $t \to S(y)_{0,t}$ both injective.
If x not reparametrisation of y, then
$\exists$ smooth functions L_1, L_2

$$\int_0^1 L_1\left(S(x)_{0,u}\right) \mathrm{d}L_2\left(S(x)_{0,u}\right) \neq \int_0^1 L_1\left(S(y)_{0,u}\right) \mathrm{d}L_2\left(S(y)_{0,u}\right.$$

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Corollary

If $t \to S(x)_{0,t}$ and $t \to S(y)_{0,t}$ injective, then

$$S(x)_{0,1} = S(y)_{0,1} \iff x \quad \text{reparametrisation of } y.$$

Isometry

Theorem (Hambly-Lyons isometry)

A path $x : [0,1] \to \mathbb{R}^d$ satisfies $\|x_t'\| = 1\ \forall t$ and is C^3, then

$$\mathit{length}\,(x) = \limsup_{n\to\infty} \left\| n! \int_{0<t_1<\ldots<t_n<1} \mathrm{d}x_{t_1} \otimes \ldots \otimes \mathrm{d}x_{t_n} \right\|^{\frac{1}{n}}, \tag{1}$$

where $\|\cdot\|$ is the projective norm w.r.t. Euclidean norm.

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Open problem

1. Is lim sup in (1) in fact a lim?
2. Extend isometry (1) to reduced bounded variation paths.

$\lim = \limsup$

Theorem (B.-Geng)

If x is a non tree-like, B.V. path, then there is at most finitely many n such that

$$\int_{0<t_1<\ldots<t_n<1} \mathrm{d}x_{t_1} \otimes \ldots \otimes \mathrm{d}x_{t_n} = 0.$$

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Theorem (Chang-Lyons-Ni)

If x is a B.V. path, then the following limit exists:

$$\lim_{n\to\infty} \left\| n! \int_{0<t_1<\ldots<t_n<1} \mathrm{d}x_{t_1} \otimes \ldots \otimes \mathrm{d}x_{t_n} \right\|^{\frac{1}{n}}.$$

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Ideas from Chang-Lyons-Ni.

If $x = (x^1, \ldots, x^d)$,

$$\int_{0<t_1<\ldots<t_n<t} \mathrm{d}x_{t_1}^{i_1} \ldots \mathrm{d}x_{t_n}^{i_n} \cdot \int_{0<t_1<\ldots<t_k<t} \mathrm{d}x_{t_1}^{i_{n+1}} \ldots \mathrm{d}x_{t_k}^{i_{n+k}}$$
$$= \sum_{\sigma\in Sh(n,k)} \int_{0<t_1<\ldots<t_{n+k}<t} \mathrm{d}x_{t_1}^{i_{\sigma^{-1}(1)}} \ldots \mathrm{d}x_{t_{n+k}}^{i_{\sigma^{-1}(n+k)}}.$$

Therefore,

$$\left|\int_{0<t_1<\ldots<t_n<t} \mathrm{d}x_{t_1}^{i_1} \ldots \mathrm{d}x_{t_n}^{i_n}\right| \cdot \left|\int_{0<t_1<\ldots<t_k<t} \mathrm{d}x_{t_1}^{i_{n+1}} \ldots \mathrm{d}x_{t_k}^{i_{n+k}}\right|$$
$$\leq |Sh(n,k)| \max_{\sigma\in Sh(n,k)} \left|\int_{0<t_1<\ldots<t_{n+k}<t} \mathrm{d}x_{t_1}^{i_{\sigma^{-1}(1)}} \ldots \mathrm{d}x_{t_{n+k}}^{i_{\sigma^{-1}(n+k)}}\right|.$$

Proof of isometry (special case)

Lemma (Hambly-Lyons10)

Let Y_t^λ be the solution to

$$\mathrm{d}Y_t^\lambda = \lambda A(\mathrm{d}x_t)\, Y_t^\lambda, \qquad Y_0^\lambda = v,$$

with $\|v\| = 1$. If $\|A\|_{\mathbb{R}^d \to L(\mathbb{R}^n, \mathbb{R}^n)} = 1$, then

$$\limsup_{\lambda \to \infty} \frac{\log \|Y_1^\lambda\|}{\lambda} \leq \lim_{n \to \infty} \left\| n! \int_{0<t_1<\ldots<t_n<1} \mathrm{d}x_{t_1} \otimes \ldots \otimes \mathrm{d}x_{t_n} \right\|^{\frac{1}{n}}.$$

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- $$Y_1^\lambda = v + \ldots + \lambda^n \int_{0<t_1<\ldots<t_n<1} A\mathrm{d}x_{t_1} \ldots A\mathrm{d}x_{t_n} v \ldots .$$

Isometry

Theorem (Lyons-Xu16)

If $\|x'\| = 1$ and x is C^1, then

$$\limsup_{n\to\infty} \left\| n! \int_{0<t_1<\ldots<t_n<1} \mathrm{d}x_{t_1} \otimes \ldots \otimes \mathrm{d}x_{t_n} \right\|^{\frac{1}{n}} = \mathit{length}\,(x)\,. \tag{2}$$

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Theorem (B.-Geng To Appear)

If $d = 2$, $\|x'\| = 1$ and

$$\forall t \in [0,1]\,, \exists \delta, a, \ s.t \ \ Arg\left(x'_s\right) \in (a, a+\pi-\varepsilon)\ \forall s \in (t-\delta, t+\delta)$$

, then (2) holds.

Isometry (rough path case)

Theorem (B.-Geng19)

If B is multi-dimensional Brownian motion, there exists deterministic $C > 0$, almost surely,

$$\limsup_{n\to\infty} \left\| \left(\frac{n}{2}\right)! \int_{0<t_1<\ldots<t_n<t} \mathrm{d}B_{t_1} \otimes \ldots \otimes \mathrm{d}B_{t_n} \right\|^{\frac{2}{n}} = Ct. \tag{3}$$

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Theorem (B.-Geng-Souris20)

If $x_t = e^{t(\mathcal{P}_1+\ldots+\mathcal{P}_M)}$, where $\mathcal{P}_i$ is Lie polynomial of degree i, there exists $C > 0$ depending on M, d,

$$C\|\mathcal{P}_M\| \leq \limsup_{n\to\infty} \left\| \left(\frac{n}{M}\right)! \int_{0<t_1<\ldots<t_n<t} \mathrm{d}x_{t_1} \otimes \ldots \otimes \mathrm{d}x_{t_n} \right\|^{\frac{M}{n}} \leq \|\mathcal{P}_M\|.$$

Image of signature

Open problem

How to characterise

$$\left\{ S\left(x\right)_{0,1} : x : [0,1] \to \mathbb{R}^{d},\ \text{continuous, BV} \right\}$$

as a subset of tensor algebra?

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Lemma (Chen)

Define $L_1 = \mathbb{R}^d$ *and if* $[a,b] = a \otimes b - b \otimes a$,

$$L_{n+1} = span\left\{ [a,b] : a \in \mathbb{R}^d, b \in L_n \right\}.$$

Then

$$\log S(x)_{0,1} \in \Pi_{n=1}^{\infty} L_n.$$

Image of signature

Lyons-Sidorova conjecture (modified)

If x has bounded variation and

$$\log S(x)_{0,1} = \sum_{n=1}^{\infty} l_n,$$

where $l_n \in L_n$, then $\exists \lambda$ such that

$$\sum_{n=1}^{\infty} \lambda^n \|l_n\| = \infty$$

unless x is conjugate and tree-like equivalent to a straight line.

Image of signature

Lemma (Key Ingredient of Lyons-Sidorova)

Lyons-Sidorova conjecture is true for the path (x, y) if there exists λ such that if

$$\mathrm{d}Y_t^\lambda = \lambda \begin{pmatrix} \mathrm{d}x_t & \mathrm{d}y_t \\ \mathrm{d}y_t & -\mathrm{d}x_t \end{pmatrix} Y_t^\lambda, \qquad Y_t^\lambda = I,$$

then $Y_1^\lambda \notin \exp\left(sl\left(2; \mathbb{R}^2\right)\right)$.

Hambly-Lyons open problems on signatures

Problem 1.10: Uniqueness problem
Problem 1.11: Inversion problem
Problem 1.12: Image problem.